A few notes on Matrix Algebra

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We'll *very* briefly explain a few aspects of linear algebra. This discussion is sufficient for the problems encountered in your homework assignments. To learn more, see any linear algebra textbook, e.g.

Howard Anton, *Elementary Linear Algebra (7th Ed.)*, John Wiley and Sons, New York, 1994.

I. MATRICES

A matrix is a rectangular array of numbers. The matrix A, below, is a 3×3 matrix, with elements a_{ij}, where the indices i and j run from 1 to 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{1}$$

The matrix B, below, is a 3×1 matrix:

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}. \tag{2}$$

The product AB is C, also a 3×1 matrix. The elements c_{i1} of C are given by

$$c_{i1} = \sum_{j=1}^{3} a_{ij} b_{j1},$$

where Σ indicates a sum. In other words, each element of C is the sum of the products of a row of A with a column of B. (Here, B only has one column.)

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \end{bmatrix}. \tag{3}$$

This is shorthand for three linear equations:

$$a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = c_{11} \tag{4}$$
$$a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = c_{21} \tag{5}$$
$$a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} = c_{31}. \tag{6}$$

When dealing with N coupled oscillators, we'll have matrix equations that look like $AB = 0$, where 0 is a $N \times 1$ array of zeros. The elements of A will involve the oscillation frequencies (ω), and the elements of B will involve the amplitudes. We'll want to find the conditions such that the equation has a solution other than the "trivial" one, $B = 0$. A fundamental theorem of linear algebra states that a non-trivial solution exists if and only if the *determinant* of A is zero. In the next section, we'll look at how to calculate a determinant.

Exercises. (Don't turn these in.)

1. Let

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & -2 & 2 \\ 2 & 1 & 5 \end{bmatrix}. \tag{7}$$

$$B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}. \tag{8}$$

Calculate $C = AB$.

Answer:

$$C = \begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix}. \tag{9}$$

2. Write the following system of equations as a matrix equation:

$$w_1C_1 + w_2C_2 + w_1C_3 = 0 \tag{10}$$
$$2w_1C_1 + 2w_1C_3 = 0 \tag{11}$$
$$w_2C_1 + w_2C_2 + 2w_1C_3 = 0. \tag{12}$$

Answer:

$$\begin{bmatrix} w_1 & w_2 & w_1 \\ 2w_1 & 0 & 2w_1 \\ w_2 & w_2 & 2w_1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}. = 0 \tag{13}$$

II. DETERMINANTS

The **determinant** of the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{14}$$

is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{15}$$

Note that this is one diagonal minus the other.

The **determinant** of the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{16}$$

is

$$\begin{aligned} \det(A) = \quad & a_{11}(a_{22}a_{33} - a_{23}a_{32}) && (17) \\ & -a_{12}(a_{21}a_{33} - a_{23}a_{31}) && (18) \\ & +a_{13}(a_{21}a_{32} - a_{22}a_{31}). && (19) \end{aligned}$$

This looks complicated, but note that we took each element of the first row of A (we could have used any row) and multiplied it by the determinant of the 2×2 matrix that is formed by excluding the row and column of the element we're considering. We then add all these terms, flipping the sign (± 1) for each alternate element. In other words:

$$\begin{aligned} \det(A) \;=\; & +a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} && (20) \\ & -a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} && (21) \\ & +a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}. && (22) \end{aligned}$$

Exercises. (Don't turn these in.)

1. Consider a "rotation matrix:"

$$R = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{23}$$

   Show that $\det(R) = 1$.

2. Evaluate the determinant of the array

$$W = \begin{bmatrix} w_1 & w_2 & w_1 \\ 2w_1 & 0 & 2w_1 \\ w_2 & w_2 & 2w_1 \end{bmatrix}. \tag{24}$$

   Answer:

$$\begin{aligned} \det(W) \;=\; & w_1(-2w_1w_2) - w_2(4w_1^2 - 2w_1w_2) && (25) \\ & +w_1(2w_1w_2) && (26) \\ =\; & -4w_1^2w_2 + 2w_1w_2^2. && (27) \end{aligned}$$